Noise-induced transitions *vs.* noise-induced phase transitions

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Abstract. I will briefly review the field of noise-induced phase transitions, emphasizing the main differences with the phase-induced transitions and showing that they appear in different systems. I will show that a noise-induced transition can disappear after a suitable change of variables and I will also discuss the breaking of ergodicity and symmetry breaking that occur in noise-induced phase transitions in the thermodynamic limit, but not in noise-induced transitions.

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BIFURCATIONS IN STOCHASTIC SYSTEMS

A bifurcation in a dynamical system is a change in the number of fixed points, or in their relative stability, that occurs when varying a control parameter, the so-called bifurcation parameter. The value of this parameter at which the change occurs is the bifurcation point [1]. The normal form of a bifurcation is the simplest mathematical model (usually involving polynomials of the lowest possible order) for which a particular change of behavior occurs. One of the simplest examples is that of the *transcritical* bifurcation for which the normal form is $dx(t)/dt = \mu x - x^2$, the Verhulst, or logistic, equation [2]. This equation can model, for instance, the growth of biological populations, or autocatalytic reactions, amongst other applications. For $\mu < 0$, there is only one (stable) fixed point at $x = 0$, whereas for $\mu > 0$ there are two fixed points: the one at $x = 0$ (which is now unstable) and another one at $x = \mu$ which is stable. Another simple example is that of the *supercritical pitchfork* bifurcation for which the normal form is $dx(t)/dt = \mu x - x^3$, the Landau equation used in the context of phase transitions in the mean-field approach. For $\mu < 0$, there is only one (stable) fixed point at $x = 0$, whereas for $\mu > 0$ there are three fixed points: the one at $x = 0$ (which is now unstable) and two more at $x = \pm\sqrt{\mu}$ which are stable. In both examples, the bifurcation point is, hence, $\mu = 0$. The importance of the stable fixed points is that, under some additional conditions, they determine the long-time dynamical behavior, as the dynamical evolution tends to one of the stable fixed points, and then it stops [3]. In the supercritical pitchfork, the value $x = +\sqrt{\mu}$ is reached if the initial condition is $x(t = 0) > 0$, whereas the fixed point at $x = -\sqrt{\mu}$ is reached whenever $x(t = 0) < 0$. The symmetry $x \to -x$ of the differential equation is broken by the initial condition in the case $\mu > 0$.

When there are stochastic, so-called noise, terms in the dynamics, usually there are no fixed points but the long-time dynamical behavior still has some preferred values. Consider, for example, the normal form for the supercritical bifurcation with an additional noise term

$$\frac{dx(t)}{dt} = \mu x - x^3 + \sqrt{2D}\xi(t), \tag{1}$$

being $\xi(t)$ a Gaussian process of zero mean and correlations $\langle \xi(t)\xi(t')\rangle = \delta(t-t')$, or *white noise* [3]. D is the noise intensity. This equation can be written in terms of relaxational dynamics [3] in a double-well potential $V(x)$:

$$\frac{dx(t)}{dt} = -\frac{\partial V(x)}{\partial x} + \sqrt{2D}\xi(t), \qquad V(x) = -\frac{\mu}{2}x^2 + \frac{1}{4}x^4. \tag{2}$$

It is possible to prove using the Fokker-Planck equation [4] (see later) that the stationary probability distribution for the x variable is $P_{\mathrm{st}}(x) = \mathscr{Z}^{-1}\exp\left[-\frac{V(x)}{D}\right]$, being $\mathscr{Z} = \int_{-\infty}^{\infty} dx \exp\left[-\frac{V(x)}{D}\right]$ a normalization factor. The stationary probability has **maxima** at $x = 0$ for $\mu < 0$ and at $x = \pm\sqrt{\mu}$ for $\mu > 0$. So it is still true that, from a probabilistic point of view, the fixed points of the deterministic, i.e. $D = 0$, dynamics are the ones **preferred** by the stochastic trajectories, but the dynamics does not end in one of the fixed points. Another important difference with the deterministic dynamics is that, for $\mu > 0$, the trajectories are not confined to the neighborhood of one of the maxima. There are constant jumps between the two maxima of the probability distribution. A classical calculation by Kramers [5], shows that the frequency of the jumps between the two maxima is proportional to $\exp\left[-\frac{\Delta V}{D}\right]$, being ΔV the height of the potential barrier between the maxima, or $\Delta V = \mu^2/4$ in the double well potential. As there are many jumps between the maxima, the noise terms have restored the symmetry $x \to -x$ of the equation.

There are other more complicated examples. Consider, for example, the Verhulst equation with the addition of a noise term ξ which is coupled multiplicatively to the dynamical variable x:

$$\frac{dx(t)}{dt} = \mu x - x^2 + \sqrt{2D}x\xi(t). \tag{3}$$

This can be thought as originated from the fact that the parameter μ randomly fluctuates and can be replaced by $\mu \to \mu + \sqrt{2D}\xi(t)$. There are some mathematical subtleties to handle the presence of the singular function $\xi(t)$. After all, the correlation function of $\xi(t)$ is a delta function, a not too well defined mathematical object. The different possible interpretations of the integral $\int dt\, g(x(t))\xi(t)$, for an arbitrary function $g(x)$, lead to different results. We will limit our considerations to the so-called Stratonovich interpretation [6, 7]. In this example, $x = 0$ is a fixed point of the stochastic dynamics. Therefore starting from $x(t=0) > 0$ as it is the case in the biological or chemical applications, the barrier $x = 0$ can never be crossed. For $\mu < 0$, the value $x = 0$ is an *attracting boundary* [6]: it will be reached in the asymptotic limit $t \to \infty$. As a consequence, the stationary probability distribution is $P_{\mathrm{st}}(x) = \delta(x)$. As μ increases and crosses zero, the picture changes. The full analysis uses the Fokker-Planck equation for the time dependent probability density $P(x,t)$. The stationary distribution for $0 < \mu < D$ is no longer a delta function at $x = 0$ but still has a maximum at $x = 0$. However, when $\mu > D$, the maximum of $P_{\mathrm{st}}(x)$ is no longer at $x = 0$ but it moves to $x = \mu - D$.

Alternatively, for fixed $\mu > 0$ one finds that the maximum of the stationary distribution switches from $x = \mu - D$ for $0 < D < \mu$ to $x = 0$ for $D > \mu$. Note that this is a somewhat counterintuitive result in the sense that a large value of the noise intensity leads to a state where the maximum of the distribution is located at a state, $x = 0$, in which the noise term $x\xi(t)$ vanishes.

Similar shifts of the maxima of the probability distribution as the noise intensity increases appear in a large class of stochastic differential equations. They have been named generically as *noise-induced transitions* [8]. In the general case of a stochastic differential equation of the form $dx(t)/dt = q(x) + \sqrt{2D}g(x)\xi(t)$, the Fokker-Planck equation reads:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[\left(q(x) - Dg(x)g'(x)\right)P(x,t)\right] + D\frac{\partial^2}{\partial x^2}\left[g(x)^2 P(x,t)\right] \tag{4}$$

and the steady-state solution $\left.\frac{\partial P(x,t)}{\partial t}\right|_{P=P_{\rm st}} = 0$ is:

$$P_{\rm st}(x) = \mathscr{Z}^{-1}\exp\left[\int^x dx'\,\frac{q(x') - Dg(x')g'(x')}{Dg^2(x')}\right]. \tag{5}$$

The maxima $\bar{x}$ of this distribution are given by

$$q(\bar{x}) - Dg(\bar{x})g'(\bar{x}) = 0. \tag{6}$$

And it is clear that $\bar{x}(D)$ depends on the noise intensity D. There are examples [8] in which equations that display the $x \to -x$ symmetry are such that for small noise intensity D the distribution is unimodal centered at $\bar{x} = 0$, and that increasing D the distribution becomes bimodal with maxima at $\pm\bar{x}(D) \neq 0$. This is the generic behavior whenever $q(x) = -x + o(x)$ and $g(x) = 1 + x^2 + o(x^2)$. A specific example is Hongler's model [9] $q(x) = -\tanh(x)$, $g(x) = \mathrm{sech}(x)$. The transition occurs at $D = D_c = 1$. The situation, in principle, could be considered the equivalent of the supercritical pitchfork bifurcation, in the sense that the most visited states are $x = 0$ for $D < 1$ and $\pm\bar{x}(D)$ for $D > 1$. However, the same remarks than in the case of the model of Eq.(1) apply: the bifurcation does not break the $x \to -x$ symmetry, as trajectories visit ergodically all possible values of x and, therefore, there are many jumps between the two preferred states. Furthermore, it is possible to show that the change in the number of maxima in the probability distribution is simply a matter of the variable used and that a simple change of variables can eliminate the bifurcation. This is explained in the next section.

NOISE-INDUCED TRANSITIONS AS A CHANGE OF VARIABLES

Let us consider the Gaussian distribution:

$$f_z(z) = \frac{1}{\sqrt{2D\pi}}e^{-z^2/2D}. \tag{7}$$

It is obviously single-peaked for all values of D, the noise intensity. Let us now introduce the new variable $x = \mathrm{argsh}(z)$ or $z = \sinh(x)$. The change of variables (i) does not depend on the noise intensity D and (ii) it is one-to-one, mapping the set of real numbers onto itself. The probability distribution for the new variable is

$$f_x(x) = f_z(z)\left|\frac{dz}{dx}\right| = f_z(z)\cosh(x), \tag{8}$$

or

$$f_x(x) = \frac{1}{\sqrt{2D\pi}}e^{-[\sinh(x)^2 - 2D\ln\cosh(x)]/2D} \equiv \frac{1}{\sqrt{2D\pi}}e^{-\frac{V_{\text{eff}}(x)}{D}}, \tag{9}$$

with an effective potential

$$V_{\text{eff}}(x) = \frac{1}{2}\sinh(x)^2 - D\ln\cosh(x), \tag{10}$$

which depends on the noise intensity. The potential is monostable for $D < D_c$ and bistable for $D > D_c$ with $D_c = 1$, as the expansion $V_{\text{eff}}(x) = \frac{1-D}{2}x^2 + \frac{2+D}{12}x^4 + O(x^6)$ shows. The Horsthemke-Lefever mechanism for noise-induced transitions is an equivalent way of reproducing this result. Just take the Langevin equation:

$$\frac{dz}{dt} = -z + \sqrt{2D}\xi(t), \tag{11}$$

being $\xi(t)$ zero-mean white noise, $\langle \xi(t)\xi(t')\rangle = \delta(t-t')$. Its steady-state probability is

$$f_z(z) = \mathscr{Z}^{-1}e^{-\frac{V(z)}{D}}, \tag{12}$$

with a potential function $V(z) = \frac{z^2}{2}$, $\mathscr{Z}$ is a normalization constant.

We now perform the aforementioned change of variables $x = \text{argsh}(z)$ to obtain (Stratonovich sense)

$$\frac{dx}{dt} = -\tanh(x) + \text{sech}(x)\sqrt{2D}\xi(t), \tag{13}$$

which is Hongler's model, one of the typical examples of noise-induced transitions explained above.

This result is very general. The same (well-known) trick can be used to reduce any one-variable Langevin equation with multiplicative noise:

$$\frac{dx}{dt} = q(x) + g(x)\sqrt{2D}\xi(t), \tag{14}$$

to one with additive noise. Simply make the change of variables defined by $dz = dx/g(x)$ or $z = \int^x dx'/g(x')$ to obtain

$$\frac{dz}{dt} = F(z) + \sqrt{2D}\xi(t), \tag{15}$$

with

$$F(z) = q(x)/g(x), \tag{16}$$

expressed in terms of the variable z. The steady-state distribution of z can be written as

$$f_z(z) = \mathscr{Z}^{-1} e^{-\frac{V(z)}{D}}, \tag{17}$$

with a potential

$$V(z) = -\int^z dz' F(z'). \tag{18}$$

The steady-state probability distribution in terms of the variable x (assuming a one-to-one change of variables) is

$$f_x(x) = f_z(z)\left|\frac{dz}{dx}\right| = \frac{f_z(z)}{|g(x)|} = \frac{\mathscr{Z}^{-1}}{|g(x)|} e^{\frac{1}{D}\int^z dz' F(z')} = \frac{\mathscr{Z}^{-1}}{|g(x)|} e^{\frac{1}{D}\int^x \frac{dx'}{g(x')}\frac{q(x')}{g(x')}} = \frac{\mathscr{Z}^{-1}}{|g(x)|} e^{\frac{1}{D}\int^x dx' \frac{q(x')}{g(x')^2}}, \tag{19}$$

the same steady-state probability distribution coming from the multiplicative-noise Langevin equation (14) that was written in Eq.(5). In terms of an effective potential: $f_x(x) = \mathscr{Z}^{-1} e^{-\frac{V\text{eff}(x)}{D}}$ we have

$$V_{\text{eff}}(x) = -\int^x dx' \frac{q(x')}{g(x')^2} + D \ln|g(x)|. \tag{20}$$

A noise-induced transition will appear if the potential $V_{\text{eff}}(x)$ changes from monostable to bistable as the noise intensity D increases.

Another widely used example of a noise-induced transition [8] is that of $q(x) = -x + \lambda x(1-x^2)$ and $g(x) = 1-x^2$. The change of variables $z = \int^x \frac{dx'}{1-x'^2} = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$, or $x = \tanh(z)$ leads to the Langevin equation:

$$\frac{dz}{dt} = -\sinh(z)\cosh(z) + \lambda \tanh(z) + \sqrt{2D}\xi(t). \tag{21}$$

Note that $x \in (-1,1)$, a fact already implied in the original Langevin equation since $x = \pm 1$ are reflecting barriers. The steady-state probability distribution of this Langevin equation is $f_z(z) = \mathscr{Z}^{-1} e^{-\frac{V(z)}{D}}$ with a potential $V(z) = \frac{1}{2}\cosh(z)^2 - \lambda \log(\cosh(z))$. The Taylor expansion $V(z) = \frac{1}{2} + \frac{1-\lambda}{2}z^2 + \frac{2+\lambda}{12}z^4 + O(z^6)$, shows that $f_z(z)$ has a single minimum at $z = 0$ for $\lambda < 1$ and double minima for $\lambda > 1$. As far as the x variable is concerned, the effective potential as given by (20) is

$$V_{\text{eff}}(x) = \frac{1}{2(1-x^2)} + \frac{\lambda + 2D}{2}\log(1-x^2). \tag{22}$$

The Taylor expansion $V_{\text{eff}}(x) = \frac{1}{2} + \frac{1-\lambda-2D}{2}x^2 + \frac{2-\lambda-2D}{4}x^4 + O[x^6]$ shows that the potential leads to a monostable distribution if $\lambda + 2D < 1$ and to a bistable one if $\lambda + 2D > 1$. Hence, a noise-induced transition occurs for $\lambda < 1$ since a bistable distribution for the x variable appears for $D > D_c = (1-\lambda)/2$. Note, however, that the distribution of the z variable is monostable for all values of D, so that the noise-induced transition is dependent on the variable considered. In the case $\lambda > 1$ the distribution is always bistable, both for the x and the z variables.

The change $x = \tanh(z)$ also induces a transition in the simpler case that the z variable follows the Gaussian distribution Eq.(7). The probability distribution function for the new variable is $q(x) = \frac{1+x^2}{\sqrt{2D\pi}} e^{-\mathrm{argth}(x)^2/2D} = \frac{1}{\sqrt{2D\pi}}\left[1 + \left(1 - \frac{1}{2D}\right)x^2 + O(x^4)\right]$ which indicates a phase transition at $D_c = 1/2$.

A remarkable example is the change $x = \frac{z}{1+|z|}$ which leads to a probability distribution $q(x) = \frac{1}{\sqrt{2D\pi}} \frac{e^{-\left(\frac{x}{1-|x|}\right)^2/2D}}{(1-|x|)^2}$ for $x \in (-1,1)$ which is bimodal for any $D > 0$, or $D_c = 0$.

NOISE-INDUCED PHASE TRANSITIONS

How can one obtain a true, symmetry breaking, bifurcation in a stochastic model? The answer lies in the coupling of many individual systems in order to obtain a bifurcation in the macroscopic variable. Let us explain this with a simple example: the standard Ginzburg-Landau model for phase transitions [10]. It consists of many coupled dynamical variables $x_i(t)$, $i = 1,\dots,N$ which individually follow Eq.(1). The full model is:

$$\frac{dx_i(t)}{dt} = \mu x_i - x_i^3 + \frac{C}{N_i}\sum_{j\in\mathcal{N}_i}(x_j - x_i) + \sqrt{2D}\xi_i(t). \tag{23}$$

The noise variables are now independent Gaussian variables of zero mean and correlations $\langle\xi_i(t)\xi_j(t')\rangle = \delta_{ij}\delta(t-t')$. $\mathcal{N}_i$ refers to the set of N_i variables x_j which are coupled to x_i. Typical situations include an all-to-all coupling where $\mathcal{N}_i$ is the set of all units and $N_i = N$, or regular d-dimensional lattices where a unit x_i is connected to the set of $N_i = 2d$ nearest neighbors, although in more recent applications one also considers non-regular, random, small world, scale free or other types of lattices [11]. C is the coupling constant. If $C = 0$ each unit is independent of the other and displays the stochastic bifurcation at $\mu = 0$ explained before. For $C > 0$, a collective state can develop in which the global variable $m(t) = N^{-1}\sum_{i=1}^N x_i(t)$ follows, in the thermodynamic limit, a true bifurcation from a state in which the stationary distribution is $P_{\mathrm{st}}(x) = \delta(m)$, to another one in which it is either $P_{\mathrm{st}}(x) = \delta(m - m_0)$ or $P_{\mathrm{st}}(x) = \delta(m + m_0)$. This is nothing but a phase transition. Here, borrowing the language from the para-ferromagnetic transition [12], m_0 is called, in this context, the *spontaneous magnetization* and it is a function of noise intensity D, coupling constant C and the parameter μ. It is important to stress that a true symmetry-breaking transition, with non-ergodic behavior, occurs only in the thermodynamic limit $N \to \infty$. For finite N the stationary probability distribution $P_{\mathrm{st}}(m)$ is either a function peaked around $m = 0$ or displays two large maxima around $\pm m_0$. The height of these maxima increases with N and the width around them decreases with N until delta-functions are reached for $N \to \infty$. One can see evidence of this behavior in Fig.1 The price one has to pay to obtain this symmetry-breaking bifurcation is that, for fixed C and D, the bifurcation point is no longer at $\mu = 0$, but is is shifted to a positive value μ_c [13]. Alternatively, for fixed $\mu > 0$ there is a bifurcation induced by varying the noise intensity: when $D < D_c$ (the critical noise intensity), the distribution of m is a delta function located either at $m = \pm m_0$; for $D > D_c$, the distribution is again a delta function around $m = 0$. The bifurcation acts in the way noise is expected to influence the dynamics: for larger noise intensity the distribution is peaked around $m = 0$ (a situation in which roughly half of the x_i variables have a positive value and the other half negative, or disordered). When the noise intensity is small, $D < D_c$, the distribution is peaked around $+m_0$ or m_0 and, hence, variables x_i have a probability distribution peaked around this value, or ordered. As either $+m_0$ or $-m_0$ is selected (depending on initial conditions and realizations of the noise variables), the $x \to -x$ symmetry has been broken for $D < D_c$ and it is restored for $D > D_c$. It is not possible, in general, to obtain the probability distribution $p(x_i,t)$ for a single unit x_i, but an approximate result can be derived within the so-called Weiss effective-field theory [14, 12]. In a nutshell, it consists in replacing the detailed interaction with the neighbors with the global variable $m(t)$. This leads to a single equation for x_i:

$$\frac{dx_i(t)}{dt} = \mu x_i - x_i^3 + C(m(t) - x_i) + \sqrt{2D}\xi_i(t). \tag{24}$$

From here it is possible to write the Fokker-Planck equation for $p(x_i,t)$. The stationary solution depends on the value of $m(t)$ in the steady state, m_0,

$$p_{\rm st}(x_i;m_0) = \mathscr{Z}^{-1}\exp\left[-v(x_i;m_0)/D\right], \qquad v(x;m) = -Cm_0x - \frac{\mu - C}{2}x^2 + \frac{1}{4}x^4 \tag{25}$$

m_0 is obtained via the self-consistency relation $\langle x\rangle = \int dx\, p_{\rm st}(x;m_0) = m_0$. This yields $m_0 = m_0(D,C,\mu)$ and it is such that, for a range of values of μ and $\mu > C$, there is a critical value D_c such that $m_0 = 0$ for $D > D_c$ and there are two solutions $\pm m_0$ with $m_0 > 0$ for $D < D_c$. Therefore, the one-unit dynamical system x_i experiences a stochastic bifurcation, in the sense that the maxima of the probability of $p_{\rm st}(x_i)$ change location as D crosses D_c.

The idea naturally arises of whether is is possible to obtain a bifurcation for the global variable if we couple N units $(x_1,x_2,\dots,x_N)$, each one of which experiences a noise-induced transition from unimodal to bimodal as the noise intensity increases. In other words, if we consider the coupled system:

$$\frac{dx_i(t)}{dt} = q(x_i) + \frac{C}{N_i}\sum_{j\in\mathscr{N}_i}(x_j - x_i) + \sqrt{2D}g(x_i)\xi_i(t). \tag{26}$$

such that the uncoupled unit $\frac{dx_i(t)}{dt} = q(x_i) + \sqrt{2D}g(x_i)\xi_i(t)$ undergoes a noise-induced transition in the sense of Hormthenske and Lefever, will the global variable $m(t)$ undergo a bifurcation from *disorder* to *order* as the noise intensity increases? The answer turns out to be no[15, 16], one of the reasons being that, as we have already noted, the shift in the maxima of the probability distribution of $p_{\rm st}(x_i)$ might disappear after a change of variables, whereas a true bifurcation remains after a one-to-one change of variables.

However, it was found quite surprisingly [15, 16] that it is possible to find functions $q(x)$ and $g(x)$ such that the global variable $m(t)$ experiences a bifurcation from $m_0 = 0$ to $\pm m_0$ with $m_0 > 0$ increasing the noise intensity D. The minimal model (normal form) is

$$\frac{dx_i(t)}{dt} = -x_i(1+x_i^2)^2 + \frac{C}{N_i}\sum_{j\in\mathscr{N}_i}(x_j - x_i) + \sqrt{2D}(1+x_i^2)\xi_i(t). \tag{27}$$

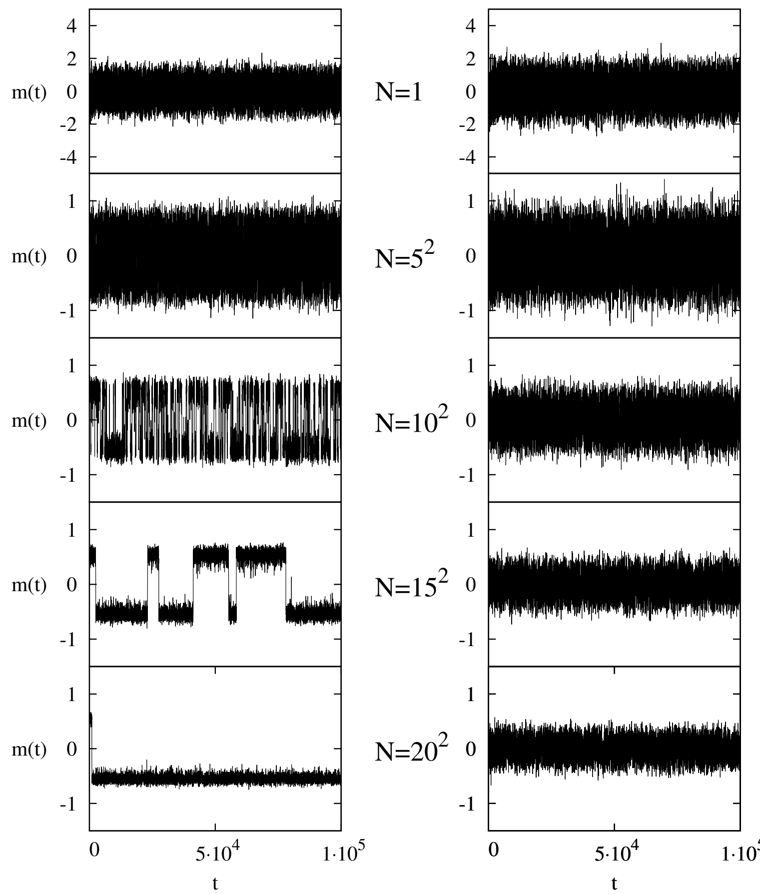

FIGURE 1. Time traces of the magnetization $m(t) = N^{-1}\sum_{i=1}^{N} x_i(t)$ for the Ginzburg-Landau model in a 2-d regular network with nearest-neighbors coupling. The right column corresponds to $D = 4$ (disordered state), and the left column to $D = 1.5$ (ordered stated). In both cases it is $\mu = 0.5$ and the coupling constant is $C = 20$. Note that the uncoupled system, $N = 1$ is always disordered as, in both cases, it has the maximum of the probability distribution located at $x = 0$. Note also that the width of the distributions decrease with N and tend to delta-functions in the limit $N \to \infty$.

It is remarkable, and counterintuitive, that a globally ordered situation arises as a result of an increase of the noise intensity. As it can be seen in Fig.2, the bifurcation is truly symmetry-breaking only for $N \to \infty$. If noise is increased even further, then a new bifurcation to the disordered state is obtained. However, as explained in detail in [15, 16] the explanation of this counterintuitive behavior has to do with the short-time dynamical instability of x_i rather than with the long-time steady distribution. We refer the interested reader to those papers and the excellent review in the book [17] for further details on this topic.

Let us now analyze this model using the results of the previous section with $q(x) = -x(1+x^2)^2$ and $g(x) = 1+x^2$. The change of variables in this case is $z = \int^x \frac{dx'}{1+x'^2} = \arctan(x)$ or $x = \tan(z)$. A one-to-one transformation is obtained if

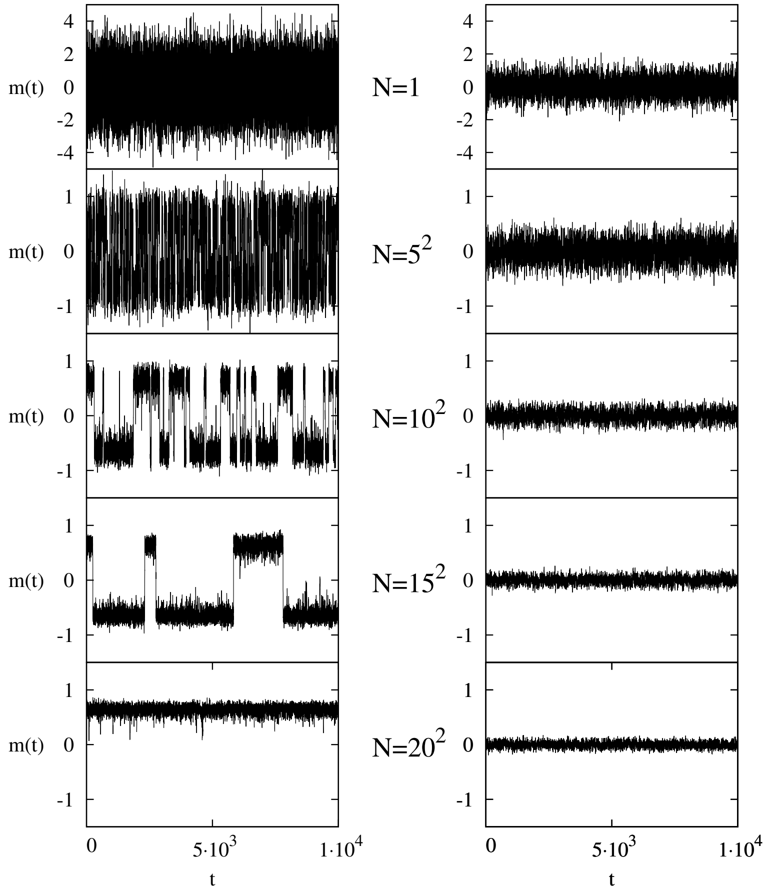

FIGURE 2. Time traces of the magnetization $m(t) = N^{-1}\sum_{i=1}^{N} x_i(t)$ for the canonical model displaying a noise-induced phase transition, Eq. (27) in a 2-d regular network with nearest-neighbors coupling. The right column corresponds to $D = 0.8$ (disorder state), and the left column to $D = 4$ (order induced by noise). The coupling constant is $C = 20$ in both cases. As in the previous figure, note that the uncoupled system, $N = 1$ is always disordered as, in both cases, it has the maximum of the probability distribution located at $x = 0$. Note also that the width of the distributions decrease with N and tend to delta-functions in the limit $N \to \infty$. Here and in Fig.1, the trajectories have been generated by a stochastic version of the Runge-Kutta algorithm, known as the Heun method [3] and using an efficient generator of Gaussian random numbers [18]

we limit $z \in (-\pi/2, \pi/2)$. The Langevin equation for the z variable is

$$\frac{dz}{dt} = -\frac{\sin(z)}{\cos(z)^3} + \sqrt{2D}\xi(t), \tag{28}$$

with a potential $V(z) = \frac{1}{2\cos(z)^2}$. The potential is monostable for $z \in (-\pi/2, \pi/2)$. The effective potential for the x variable is:

$$V_{\text{eff}}(x) = \frac{x^2}{2} + D\log(1+x^2) \tag{29}$$

which, again, is always monostable. Therefore, in this case the change of variables does not induce any bistability.

In summary, we have revisited the concept of noise-induced transitions, defined as shifts in the maxima of the steady state probability distribution. They can not be considered "*bona fide*" bifurcations in the standard sense as (i) they can disappear through a one-to-one change of variables and (ii) there is no true symmetry breaking as all states can be visited independently of the initial condition. A noise-induced phase transition, on the other hand, can appear in the global variable of a coupled system. In the thermodynamic limit it displays symmetry breaking and lack of ergodicity. There are bifurcations from disorder to order when increasing the noise intensity (as in the Ginzburg-Landau model) but, more remarkably, there are cases in which an ordered state can appear as a result of an increase of the noise intensity. Generally, the transition is reentrant, in the sense that a large noise recovers the ordered state, but it is possible to find other situations in which reentrance does not occur [19].

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